DEFORMATIONS OF VECTOR BUNDLES ON COISOTROPIC SUBVARIETIES VIA THE ATIYAH CLASS

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Abstract. Using the Atiyah class we give a criterion for a vector bundle on a coisotropic subvariety, Y, of an algebraic Poisson variety X to admit a first and second order noncommutative deformation. We also show noncommutative deformations of a vector bundle are governed by a curved dg Lie algebra which reduces to the classical relative Hochschild complex when the Poisson structure on X is trivial.

1. Introduction

Let X be a smooth algebraic variety and let $\mathcal{O}_X$ denote the sheaf of regular functions on X. Recall a deformation quantization of order 2 of X is a flat sheaf $\mathcal{A}_2$ of algebras over $k[\epsilon]/\epsilon^3$ such that on affine subsets U_i of a Zariski open covering of X we have $\mathcal{A}_2|_{U_i} \simeq (\mathcal{O}_X \oplus \epsilon\mathcal{O}_X \oplus \epsilon^2\mathcal{O}_X)|_{U_i}$ as sheaves of $k[\epsilon]/\epsilon^3$-modules. The product is locally given by

$$a *_i b = ab + \epsilon\alpha_1^{Xi}(a,b) + \epsilon^2\alpha_2^{Xi}(a,b)$$

where a, b are local section of $\mathcal{O}_{U_i}$ and $\alpha_1^{Xi}(a,b) = \frac{1}{2}P(da, db)$ for a globally defined bivector $P \in H^0(X, \wedge^2 T_X)$ and α_2^{Xi} is a bidifferential operator. On double intersection $U_i \cap U_j$ the restrictions are identified by sending a regular function f to $f + \epsilon\beta_1^{Xij}(f) + \epsilon^2\beta_2^{Xij}(f)$ where $\beta_1^{Xij}, \beta_2^{Xij}$ are differential operators from $\mathcal{O}_{U_i\cap U_j}$ to $\mathcal{O}_{U_i\cap U_j}$. In a far fancier language than we will need here $U \mapsto \mathcal{A}_2(U)$ is a presheaf of algebroids [18].

In this paper we consider the higher rank version of [4]: let $Y \subset X$ be a smooth closed coisotropic subvariety of a smooth Poisson variety X and E a vector bundle on Y. Viewing E as coherent $\mathcal{O}_X$-module a natural question to ask is when does E admit a flat second order deformation to an $\mathcal{A}_2$-module. This means, we want a coherent sheaf $\mathcal{E}_2$ which splits locally on an affine open cover $\{U_i\}$, with a module action given by

$$a *_i e = ae + \epsilon\alpha_1^i(a,e) + \epsilon^2\alpha_2^i(a,e)$$

and transition functions on $U_i \cap U_j$ given by

$$e \mapsto e + \epsilon\beta_1^{ij}(e) + \epsilon^2\beta_2^{ij}(e)$$

where a, e are local sections of $\mathcal{O}_X$, and E, respectively, and α_1^i, α_2^i and $\beta_1^{ij}, \beta_2^{ij}$ are (bi)differential operators.

For simplicity we set $F(E) := F \otimes_{\mathcal{O}_Y} End_{\mathcal{O}_Y}(E)$ where F is a coherent sheaf on X. Using spectral sequences there are three obstructions to the existence of $\mathcal{E}_1$ in $H^0(Y, \wedge^2 N(E))$, $H^1(Y, N(E))$ and $H^2(Y, \mathcal{O}_Y(E))$ where N is the normal bundle of Y in X. The first obstruction measures whether $\mathcal{E}_1$ exists locally in the Zarkiski/étale topology. If an infinitesimal deformation exists locally the class

in $H^1(Y, N(E))$ is well-defined and its vanishing is equivalent to the existence of transition functions β_1^{ij} which agree with the module structure. The class in $H^2(Y, \mathcal{O}_Y(E))$ is well-defined when the previous class vanishes and this class vanishes precisely when the transition functions satisfy the cocycle condition on each triple intersection $U_i \cap U_j \cap U_k$. The class in $H^2(Y, \mathcal{O}_Y(E))$ for $Y = X$ has been studied in [5]. When $Y \subset X$ the author believes it is connected to Rozanksy-Witten invariants but we leave this for future study cf. [6].

In [3] it was shown the first obstruction class is the image of P in $H^0(Y, \wedge^2 N(E))$ and its vanishing is equivalent to Y be coisotropic cf. Lemma 2.2. Recall, that Y is coisotropic if $P(I_Y, I_Y) \subset I_Y$ where I_Y is the sheaf of regular functions vanishing on Y. With this in mind, we now assume that Y is *coisotropic*. By coisotropness of Y the bivector P defines a morphism $p : N^\vee \to T_Y$ along with its adjoint $p^* : \Omega^1_Y \to N$. Throughout the paper we fix a line bundle L which admits a first/second order deformation. In the case when $\beta_1^X \equiv 0$ and α_2^X is symmetric we can take $L = (\det N)^{1/2}$ [4].

Denote by

$$at_N(E \otimes_{\mathcal{O}_Y} L^\vee) := p^*(at(E \otimes_{\mathcal{O}_Y} L^\vee))$$

the Yoneda product of p^* and $at(E \otimes L^\vee) \in H^1(Y, \Omega^1_Y(E))$. Where $at(M)$ is the Atiyah class of a vector bundle M [2].

Theorem 1.1. *Let X be a smooth algebraic variety with a bivector P and Y a smooth coisotropic subvariety with a vector bundle E. If E admits a first order deformation $\mathcal{E}_1$ then*

$$at_N(E \otimes_{\mathcal{O}_Y} L^\vee) = 0$$

in $H^1(Y, N(E))$. If, in addition, $H^2(Y, \mathcal{O}_Y(E)) = 0$ the above equality in $H^1(Y, N(E))$ is also sufficient for the existence of a first order deformation. In particular, a first order deformation exists when X and Y are affine. Moreover, in the affine case there is a globally split deformation, i.e. $\mathcal{E}_1 \simeq E \oplus \epsilon E$ as sheaves of $k[\epsilon]/\epsilon^2$-modules.

A first order deformation up to isomorphism is given by a collection of operators $\gamma_E^i : N^\vee \to \mathcal{D}^1_\heartsuit(E)$ on a Zariski open cover $\{U_i\}$ which satisfy a gluing condition on $U_i \cap U_j$ cf. Proposition 2.1; $\mathcal{D}^1_\heartsuit(E)$ are first order differential operators with scalar principal symbol. Using this we give an explicit connection on $E \otimes L^\vee$ which represents the class in Theorem 1.1.

With regards to second order deformations we tacitly assume $\beta_1^X \equiv 0$. For $\mathcal{A}_2$ a second order deformation $\{a, b\}_P := 2\alpha_1^X(a, b)$ is a Lie bracket. By coisotropness of Y the conormal bundle $N^\vee = I_Y/I_Y^2$ becomes a Lie algebra where I_Y is the ideal of functions that vanish on Y. By Proposition 2.1 a first order deformation gives a global operator γ which defines a morphism between Lie algebras which will not respect the bracket in general (we are using the assumption that $\beta_1^X \equiv 0$). The *curvature*, $c(\gamma)$, measures the failure of γ to be a morphism of Lie algberas. We define the *normal complex of E* as

$$\mathcal{N}_E^\bullet : \left\{ 0 \to \mathcal{O}_Y(E) \overset{d_{N_E}}{\to} N(E) \overset{d_{N_E}}{\to} \wedge^2 N(E) \overset{d_{N_E}}{\to} \cdots \right\} \tag{1.1}$$

where the odd derivation is given by

$$d_{N_E}\omega(x_0,\ldots,x_{n+1}) = \sum_{j=0}^{n+1}(-1)^j[\gamma(x_j,\cdot),\omega(x_0,\ldots,\widehat{x}_j,\ldots,x_{n+1})] + \sum_{i<j}(-1)^{i+j}\omega(\{x_i,x_j\}_P,x_0,\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_{n+1})$$

and the x_i's are local sections of $N^\vee$. A straightforward but tedious calculation shows $c(\gamma) \in H^0(Y,\wedge^2 N(E))$. Another standard computation using the Jacobi identity for $\{,\}_P$ gives $d^2_{N_E}\omega = [c(\gamma),\omega]$ and $d_{N_E}c(\gamma) = 0$ (second Bianchi identity) which make $\mathcal{N}_E$ into a curved dg Lie algebra [20]. When $\mathcal{E}_2$ exists $\mathcal{N}_E$ is *weakly obstructed* meaning $[c(\gamma),\omega] = 0$ for all ω. In the case when E is rank 1 the complex is automatically "weakly obstructed."

For ease of the notation we make a definition similar to Deligne's λ-*connections*.

Definition 1.2. A (λ,μ)-*connection* on a vector bundle E is a k-linear operator $\nabla : N^\vee \to \mathcal{D}^1_\heartsuit(E)$ whose principal symbols are

$$\nabla(ax,e) - a\nabla(x,e) = \lambda p(x)(a)e; \qquad \nabla(x,ae) - a\nabla(x,e) = \mu p(x)(a)e$$

We denote the set of (λ,μ)-connections by $\mathcal{D}^1_{(\lambda,\mu)}(E)$. A $(0,1)$-connection is an $N^\vee$-connection from [4].

Theorem 1.3. *Assume E admits a second order deformation $\mathcal{E}_2$ and $\beta_1^X \equiv 0$ then $c(\nabla) = 0$ where ∇ is the $(0,1)$-connection on $E \otimes L^\vee$ given by the first order deformation. If $H^1(Y,N(E)) = H^2(Y,\mathcal{O}_Y(E)) = 0$ this equality also implies the existence of a second order deformation. For affine X and Y the deformation of E may be chosen globally split: $\mathcal{E}_2 \simeq E \oplus \epsilon E \oplus \epsilon^2 E$ as sheaves of $k[\epsilon]/\epsilon^3$-modules.*

The outline of the paper is as follows: In section 2 we give the proofs of Theorems 1.1 and 1.2. Section 3 we show a flat bundle can be deformed to second order after twisting by a line bundle which admits a second order deformation. The last section shows deforming a module is not governed by dg Lie algebra but a curved dg Lie algebra defined over ring of formal power series. We have also included an appendix with the module equations to order 2 and a version of the HKR theorem which we use throughout the paper.

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2. First and second order deformations

2.1. **First Order.** Suppose there is a first order deformation $\mathcal{E}_1$ of a vector bundle E. This means there is an affine open cover $\{U_i\}$ of X such that $\mathcal{E}_1 \simeq (E \oplus \epsilon E)|_{U_i}$ for all i as sheaves of $k[\epsilon]/\epsilon^2$-modules. In particular by (A.5) the operator α_1^i vanishes on $I_Y^2 \otimes E$. Denote by γ_E^i the restriction of α_1^i to $N^\vee \otimes E \to E$ which we view as a bidifferential operator on $Y \cap U_i$. Applying (A.5) twice implies γ_E^i is a $(1/2,1)$-connection. On double intersections (A.7) reduces to

$$\gamma_E^j(x,e) - \gamma_E^i(x,e) + \beta_1^{Xij}(x)e = 0 \tag{2.1}$$

The following proposition proven in [4] determines a first order deformation up to isomorphism.

Proposition 2.1. *The collection of $(1/2,1)$-connections $\{\gamma^i_E\}$ defines $\mathcal{E}_1$ uniquely up to isomorphism if $H^1(Y,\mathcal{O}_Y(E))=0$. Conversely, if Y satisfies $H^2(Y,\mathcal{O}_Y(E))=0$, for any such collection of $(1/2,1)$-connections satisfying (2.1), there exists a first order deformation $\mathcal{E}_1$ inducing it.*

A local first order deformation exists if and only if P projects to zero in $H^0(Y,\wedge^2 N(E))$ which is a priori weaker than being coisotropic. However, an easy application of the HKR theorem shows a first order deformation locally exists if and only if Y is coisotropic.

Lemma 2.2. *The projection of P is contained in $H^0(Y,\wedge^2 N)\subset H^0(Y,\wedge^2 N(E))$. Locally there exists a first order deformation if and only if the projection of P vanishes, that is Y is coisotropic.*

Proof. The following proof is an application Lemma A.1, which will be used repeatedly, so we will give all the details. A first order deformation locally exists if and only if there is an α^i_1 which satisfies (A.5). Applying Lemma A.1 we must show that $G(a,b,e):=\alpha^X_1(a,b)e$ is symmetric when restricted to $I\otimes I\otimes E\to E$ and is a cocycle i.e.

$$aG(b,c,e)-G(ab,c,e)+G(a,bc,e)-G(a,b,ce)=0$$

This holds since α^X_1 is a cocycle in $C^*(\mathcal{O}_X,\mathcal{O}_X)$ [19]. The projection of P in $H^0(Y,\wedge^2 N(E))$ is the anti-symmetrization of $G(x,y,e)$ for $x,y\in I_Y$. This is a scalar endomorphism since $G(a,b,e)$ is a scalar endomorphism. The cocycle α^X_1 comes from the Poisson structure hence is antisymmetric. Since a cocycle is symmetric if and only if the anti-symmetrization vanishes we must have $2\alpha^X_1(x,y)e=0$ for all $x,y\in I$ and $e\in E$. This implies $\alpha^X_1(x,y)\in I$ for all $x,y\in I$ which is the coisotropic condition. □

We first need a couple of lemmas which will be useful in the proof of Theorem 1.1 and later in the text.

Lemma 2.3. *Let E,F be two vector bundles on Y with connections $\gamma_E\in D_{(\lambda,\mu)}(E)$ and $\gamma_F\in\mathcal{D}_{(\lambda',\mu)}(F)$. Then*

$$\gamma_{E\otimes F}(x,e\otimes f):=\gamma_E(x,e)\otimes f+e\otimes\gamma_F(x,f)\tag{2.2}$$

defines an element of $\mathcal{D}_{(\lambda+\lambda',\mu)}(E\otimes_{\mathcal{O}_Y}F)$. The curvature of $\gamma_{E\otimes F}$ is given by

$$c(\gamma_{E\otimes F})(x,y)(e\otimes f)=c(\gamma_E)(x,y)(e)\otimes f+e\otimes c(\gamma_F)(x,y)(f)$$

Proof. The proof is by direct calculation: let $a\in\mathcal{O}_Y$, $x\in N^\vee$ then

$$\begin{aligned}\gamma_{E\otimes F}(ax,e\otimes f)-a\gamma_{E\otimes F}(x,e\otimes f)&=\gamma_E(ax,e)\otimes f+e\otimes\gamma_F(ax,f)-a\gamma_E(x,e)\otimes f-ae\otimes\gamma_F(x,f)\\&=\lambda p(x)(a)e\otimes f+\lambda'p(x)(a)e\otimes f\\&=(\lambda+\lambda')p(x)(a)e\otimes f\end{aligned}$$

The other symbol gives

$$\begin{aligned}\gamma_{E\otimes F}(x,ae\otimes f)-a\gamma_{E\otimes F}(x,e\otimes f)&=\gamma(x,ae)\otimes f+ae\otimes\gamma_F(x,f)-a\gamma_E(x,e)\otimes f-ae\otimes\gamma_F(x,f)\\&=\mu p(x)(a)e\otimes f\end{aligned}$$

The curvature formula is standard from differential geometry. □

Lemma 2.4. *Let L be as above then there exists a collection $\{\gamma^i_{L^\vee}\}$ of $(-1/2,1)$-connections on $L^\vee$ such that on double intersections we have $\gamma^i_{L^\vee}(x,l^\vee)-\gamma^j_{L^\vee}(x,l^\vee)-\beta^{Xij}_1(x)l^\vee=0$.*

Proof. Fix a section l of L and let $l^\vee$ be the dual section of $L^\vee$ under the non-degenerate $\mathcal{O}_Y$-bilinear pairing $\langle \cdot, \cdot \rangle : L \otimes_{\mathcal{O}_Y} L^\vee \to \mathcal{O}_Y$. Define an operator on $L^\vee$ via the Leibniz rule

$$\partial_x(\langle l, l^\vee \rangle) = \gamma_L(x, l) \otimes l^\vee + l \otimes \gamma_{L^\vee}(x, l^\vee) \tag{2.3}$$

By Lemma 2.3, $\gamma_{L^\vee}$ is a $(-1/2, 1)$-connection. The formula on double intersections holds since the left hand side is a global connection. □

Proof of theorem 1.1. Suppose γ_E^i exists and on U_i we define $\nabla^i : N^\vee \to \mathcal{D}^1_\heartsuit(E \otimes L^\vee)$ by

$$\nabla^i(x, e \otimes l^\vee) = \gamma_E^i(x, e) \otimes l^\vee + e \otimes \gamma_{L^\vee}^i(x, l^\vee)$$

which is $(0,1)$-connection by the previous two lemmas. It is also easy to check on double intersections that $\nabla^i - \nabla^j = 0$. The cocycle $\nabla^i - \nabla^j$ represents the class $at_N(E \otimes L^\vee) \in H^1(Y, N(E))$. Since the connections ∇^i are chosen up to a section of $H^0(U_i, N(E))$ we see that

$$at_N(E \otimes_{\mathcal{O}_Y} L^\vee) = 0$$

Conversely, if the equality holds, we can find $(0,1)$-connections ∇^i on U_i which glue to a global connection. We now define γ_E^i to be

$$\gamma_E^i(x, e) = \frac{\nabla^i(x, e \otimes l^\vee) - e \otimes \gamma_{L^\vee}^i(x, l^\vee)}{l^\vee}$$

where $l^\vee$ is any local section of $L^\vee$. We now apply the previous proposition. □

Remark. In the case when $H^2(Y, \mathcal{O}_Y(E)) = 0$ there is a bijection

$$\{\text{first order deformations of } E,\, \mathcal{E}_1\} / \sim \leftrightarrow \left\{(0,1)\text{-connections on } E \otimes_{\mathcal{O}_Y} L^\vee\right\}$$

To any $(0,1)$-connection on $E \otimes_{\mathcal{O}_Y} L^\vee$ there is a collection $(1/2, 1)$-connection on E by Lemma 2.3 which satisfy (2.1). Applying Lemma 2.1 shows there is a bijection. There is a $(0,1)$-connection when $at_N(E \otimes_{\mathcal{O}_Y} L^\vee) = 0$ in $H^1(Y, N(E))$.

Proposition 2.5. *Given a first order deformation $\mathcal{E}_1$, its group of automorphisms restricting to the identity modulo ϵ is isomorphic to $H^0(Y, \mathcal{O}_Y(E))$. If $H^1(Y, \mathcal{O}_Y(E)) = H^2(Y, \mathcal{O}_Y(E)) = 0$ and the condition imposed on $at(E \otimes_{\mathcal{O}_Y} L^\vee)$ holds, then the set of isomorphism classes of first order deformations is a torsor over $H^0(Y, N(E))$.*

Proof. Let $\phi : \mathcal{E}_1 \to \mathcal{E}_1$ be an automorphism which restricts to the identity modulo ϵ then $1 - \phi$ takes values in $\epsilon\mathcal{E}_1$ and hence descends to $\mathcal{E}_1/\epsilon\mathcal{E}_1 \simeq E \to E \simeq \epsilon E$. The map $1 - \phi$ gives a section $\phi_1 \in H^0(Y, \mathcal{O}_Y(E))$ since it is $\mathcal{O}_Y$-linear. Therefore $\phi = 1 + \epsilon\phi_1$.

By Proposition 2.1 the vanishing of the cohomology groups implies the isomorphism class is uniquely determined by the choice of γ_i. The difference of two $(1/2, 1)$-connections will be an $\mathcal{O}_Y$-bilinear map $N^\vee \times E \to E$. This means the difference will be a section of $N(E)$ over U_i. Moreover, equation (2.1) shows that such a section will glue on $U_i \cap U_j$. □

2.2. **Second Order.** In this subsection assume that $\beta_1^X \equiv 0$. When $\mathcal{A}_2$ exists locally there are bidifferential operators α_2^{Xi} which satisfy (A.2) along with gluing conditions on double intersections (A.4). By skew-symmetry of α_1^X and (A.2) the anti-symmetrization $\mathscr{A}_2^i(a,b) := \alpha_2^{Xi}(a,b) - \alpha_2^{Xi}(b,a)$ satisfies

$$a\mathscr{A}_2^i(b,c) - \mathscr{A}_2^i(ab,c) + \mathscr{A}_2^i(a,bc) - \mathscr{A}_2^i(a,b)c = 0$$

The HKR isomorphism shows $\mathscr{A}_2^i$ is given by a bivector in $H^0(U_i, \wedge^2 T_X)$. Since, $\beta_1^X \equiv 0$ the collection $\{\mathscr{A}_2^i\}$ glues to a global bivector $\mathscr{A}_2 \in H^0(X, \wedge^2 T_X)$.

Proof of theorem 1.3. Recall, we now assume $\beta_1^X \equiv 0$. Let X be affine and suppose we are given a first order deformation $\mathcal{E}_1 \simeq E \oplus \epsilon E$ from the previous theorem. To extend to $\mathcal{E}_2$ we need to find α_2 which solves (A.6). By Lemma A.1 the existence of α_2 is equivalent to the vanishing of the antisymmetrization of (A.6) when restricted to I_Y i.e. we must solve

$$\mathscr{A}_2(x,y)e = \gamma(x, \gamma(y,e)) - \gamma(y, \gamma(x,e)) - \gamma(\{x,y\}_P, e)$$

By assumption, L has a second order deformation which implies $c(\gamma_L)(x,y)(l) = \mathscr{A}_2(x,y)l$ by the previous paragraph. The left hand side of (2.3) is a flat connection therefore $c(\gamma_{L^\vee})(x,y)(l^\vee) = -\mathscr{A}_2(x,y)l^\vee$. Using Lemma 2.3 we see $c(\nabla) = 0$.

In general, the same reasoning implies the existence of operators $\alpha_2^i(a,e)$ on affine subsets U_i. By Lemma A.1 the existence of β_2^{ij} satisfying (A.8) is equivalent to

$$\alpha_2^j(x,e) - \alpha_2^i(x,e) + \beta_2^{Xij}(x)e + \alpha_1^j(x, \beta_1^{ij}(e)) - \beta_1^{ij}(\alpha_1^i(x,e)) = 0 \tag{2.4}$$

A straightforward calculation shows the RHS of (A.8) is a Hochschild cocycle and hence $\mathcal{O}_X$-bilinear when restricted to $I \otimes E$ by lemma A.2. Moreover, it vanishes for $x \in I_Y^2$ and therefore descends to $N^\vee$ defining a section of $c_{ij} \in H^0(U_i \cap U_j, N(E))$. If this class vanishes in $H^1(Y, N(E))$ then there are sections $c_i \in H^0(U_i, N(E))$ such that $c_{ij} = c_i - c_j$ on $U_i \cap U_j$. For fixed i, the conormal sequence

$$0 \to N^\vee \to \Omega_X^1|_Y \to \Omega_Y^1 \to 0$$

splits since $U_i \cap Y$ is affine and the three sheaves in the sequence are locally free. Denote the surjection by $\pi_i : \Omega_X^1|_Y \to N^\vee$. The expression, $c_i(x)e$ can then be lifted to an operator $\psi_i(a,e) = c_i(\pi_i(da))e$ which is a Hochschild cocycle. We now replace $\alpha_i(a,e)$ with $\alpha_i(a,e) - \psi_i(a,e)$ to ensure (A.8) holds. Since $H^2(Y, \mathcal{O}_Y(E)) = 0$ we can adjust β_2^{ij} by adding $\mathcal{O}_Y$ linear operators $E \to E$ on $U_i \cap U_j$ so the cocycle condition for gluing function holds on $U_i \cap U_j \cap U_k$. This completes the proof. $\square$

Remark 2.6. If X is affine and α_2^X is symmetric the two previous theorems imply any vector bundle supported on a coisotropic subvariety along with a flat connection along the *null foliation* has a second order quantization. When the subvariety, Y, is Lagrangian any vector bundle on Y with a flat $(0,1)$-connection has a second order quantization.

Proposition 2.7. *Assume $H^1(Y, \mathcal{O}_Y(E)) = 0$. (a) Let E be a vector bundle which admits a second order deformation $\mathcal{E}_2$ and let $\phi : \mathcal{E}_1 \to \mathcal{E}_1$ be an automorphism restricting to the identity modulo ϵ. If $\phi_1 \in H^0(Y, \mathcal{O}_Y(E))$ is the regular section corresponding to ϕ via Proposition 2.5 then ϕ extends to a second order automorphism $\phi_2 : \mathcal{E}_2 \to \mathcal{E}_2$ if and only if $d_{N_E}\phi_1 = 0$. In this case the set of all extensions ϕ_2 is a torsor over $H^0(Y, \mathcal{O}_Y(E))$.*

(b) Assume that E has two second order deformations $\mathcal{E}_2$ and $\mathcal{E}_2'$ such that for their first order truncations we have

$$\mathcal{E}_1' = \mathcal{E}_1 + \zeta; \qquad \zeta \in H^0(Y, N(E))$$

in the sense of the torsor structure of Proposition 2.5. Then $d_{N_E}\zeta + [\zeta,\zeta] = 0$ *in* $H^0(Y, \wedge^2 N(E))$. *If* $H^2(Y, \mathcal{O}_Y(E)) = H^1(Y, N(E)) = 0$ *and* ζ *satisfying* $d_{N_E}\zeta + [\zeta,\zeta] = 0$ *is fixed, then for a given* $\mathcal{E}_2$ *and* $\mathcal{E}_1'$ *the set of isomorphism classes of* $\mathcal{E}_2'$ *is a torsor over* $H^0(Y, N(E))$.

Proof. Locally the automorphism is of the form $e \mapsto e + \epsilon\phi_1(e) + \epsilon^2\eta(e)$. If we write out the equation $\phi(a \star e) = a \star \phi(e)$ then we see that

$$\eta(ae) - a\eta(e) = \alpha_1(a, \phi_1(e)) - \phi_1(\alpha_1(a, e))$$

By Lemma A.1 such an η exists if and only if the RHS vanishes for $a \in I_Y$. This is precisely the condition $d_{N_E}\phi_1 = 0$.

We now show that two open sets U_i and U_j are related by the transition $e \mapsto e + \epsilon\beta_1^{ij}(e) + \epsilon^2\beta_2^{ij}(e)$. This leads to

$$\eta_i(e) - \eta_j(e) - \beta_1^{ij}(\phi_1(e)) + \phi_1(\beta_1^{ij}(e)) = 0$$

which may not hold with the original η_i, η_j but these may be adjusted by an $\mathcal{O}_Y$-linear endomorphism of E on U_i and U_j. The defining equations for η, equation (A.7), and that ϕ_1 is an $\mathcal{O}_Y$-linear endomorphism imply the LHS is $\mathcal{O}_Y$-linear and defines a cocycle in $H^1(Y, \mathcal{O}_Y(E))$. Since $H^1(Y, \mathcal{O}_Y(E)) = 0$ the η_i's can adjusted to ensure the local automorphisms agree on double intersections. The only remaining ambiguity for η_i is the addition of a globally defined $\mathcal{O}_Y$-linear endomorphism of E.

To prove (b) we recall that if $H^1(Y, N(E)) = 0$ then a first order deformation is determined by a collection of $(1/2, 1)$-connections. Given two second order deformations $\mathcal{E}_2$ and $\mathcal{E}_2'$ whose first order deformations satisfy

$$\gamma_E^i(x, e) - \gamma_E^{i'}(x, e) = \zeta(x)e$$

for some $\zeta \in H^0(Y, N(E))$ implies $c(\gamma) = c(\gamma' + \zeta)$. A quick calculation shows $c(\gamma' + \zeta) = c(\gamma') + d_{\mathcal{N}_E}\zeta + [\zeta,\zeta]$. Since γ and γ' extend to second order theorem 1.3 shows their curvatures are equal $c(\gamma) = c(\gamma')$.

Conversely, if $d_{\mathcal{N}_E}\zeta + [\zeta,\zeta] = 0$ then the above calculation shows that $\gamma_E^i(x, e) + \zeta(x)e$ satisfies the curvature equation of Theorem 1.3. Hence there exists local operators $\alpha_2^i(x, e)$ satisfying (A.6). The assumptions $H^2(Y, \mathcal{O}_Y(E)) = H^1(Y, N(E)) = 0$ imply that all obstructions to existence of β_2^{ij} which satisfy (A.8) vanish. The second order deformation corresponding to ζ can be found. □

3. Deforming flat vector bundles

Throughout this section we will assume that X, Y are affine varieties. The statements can be generalized to the non-affine case with suitable cohomology vanishing which we leave to the motivated reader. Furthermore, we also assume that P is non-degernate i.e. symplectic. In this case $p : N^\vee \to T_Y$ is an embedding of vector bundles. The image, T_F, is the *null-foliation* of Y.

3.1. **Flat vector bundles.** The main result of this section is the following theorem:

Theorem 3.1. *There is a bijection*

$$\mathcal{M}_F(Y) \underset{dequant}{\overset{quant}{\rightleftarrows}} \mathcal{Q}_2(Y)$$

where $\mathcal{M}_F(Y)$ *is the set of vector bundles on* Y *which admit a* $(0,1)$*-connection flat along the null foliation and* $\mathcal{C}_2(Y)$ *is the set of vector bundles on* Y *which admit a second order deformation.*

Proof. For M a vector bundle with a connection ∇_M that is flat along T_F let $quant(M) := M \otimes_{\mathcal{O}_Y} L$. Define a $(1/2,1)$-connection on $M \otimes_{\mathcal{O}_Y} L$ via (2.2). Therefore by Proposition 2.1 $M \otimes_{\mathcal{O}_Y} L$ admits a first order deformation. By Lemma 2.3 $c(\gamma_M) = \mathscr{A}_2$ then Theorem 1.3 shows $M \otimes L$ admits a second order deformation.

If M is a bundle which admits a second order deformation then

$$\nabla_{M\otimes L^\vee}(x, m \otimes l^\vee) = \gamma_M(x,m) \otimes l^\vee + m \otimes \gamma_{L^\vee}(x, l^\vee)$$

is a flat $(0,1)$-connection on $dequant(M) := M \otimes_{\mathcal{O}_Y} L^\vee$ again by Lemma 2.3. □

Remark 3.2. Let X be a smooth variety then to any $\mathcal{D}$-module one can associate a coisotropic subvariety $Y \subset T_X$ i.e. the singular support. Let $W \subset X$ be a smooth subvariety with a local system which we view as a coherent $\mathcal{D}$-module on X via the direct image. The singular support is then $N^*_{X/W} \subset T^*_X$ which is a Lagrangian subvariety with a local system induced by the local system on W. Denote by $\pi : T^*_X \to X$ the projection map. Using the sequence

$$0 \to \pi^* T^*_W \to N_{T^*_X/W} \to \pi^* N_{X/W} \to 0$$

we see that $\wedge^* N_{T^*_X/W}$ has a second order deformation. By the above theorem, the local system on $N_{X/W}$ can be deformed to second order over the deformation quantization of $\mathcal{O}_{T^*X}$ given by the standard symplectic form on T_X after twisting by $\wedge^* N_{T^*_X/W}$.

In unpublished work, Dmitry Kaledin has proven the same theorem for smooth $\mathcal{D}$-modules with smooth support but for infinite order deformations using completely different methods. [16].

A direct corollary of the above proof shows that $\mathcal{Q}_i(Y)$ is a symmetric monoidal category

Corollary 3.3. *If $Y \subset X$ are affine then the category of second order deformations, $\mathcal{Q}_2(Y)$ is a symmetric monoidal category.*

Proof. Define a tensor product via

$$\begin{aligned} \boxtimes : \mathcal{Q}_1(Y) \times \mathcal{Q}_1(Y) &\to \mathcal{Q}_1(Y) \\ (E_1, E_2) &\to E_1 \otimes_{\mathcal{O}_Y} E_2 \otimes_{\mathcal{O}_Y} L^\vee \end{aligned}$$

The identity element is given by L. It is then clear $E_1 \boxtimes E_2$ admits a first/second order deformation with the above hypotheses. Moreover, it is easy to check that $\boxtimes$ is symmetric and associative using Lemma 2.3. □

In the case when Y is lagrangian i.e. $\dim Y = 1/2 \dim X$, there is an isomorphism $p : N^\vee \simeq T_Y$. To any vector bundle with a flat connection there corresponds module over $\mathcal{A}_2$ which splits as sheaf of $k[\epsilon]/\epsilon^3$-modules. When α_2^X is symmetric we can take $L = (\det N)^{1/2} = K_Y^{1/2}$ if it exists. The quantization map is given by twisting by $K_Y^{1/2}$.

3.2. **Atiyah algebra.** Define the *null foliation Atiyah algebra* $At_F(E) \subset \mathcal{D}(E)$ to be those operators operators whose symbol belongs to $T_F \subset End_{\mathcal{O}_Y}(E) \otimes_{\mathcal{O}_Y} T_F$. By coisotropness $At_F(E)$ is a Lie algebra since T_F is involutive. We can also define $At_F(E)$ by the following *null foliation Atiyah sequence*

$$0 \to End_{\mathcal{O}_Y}(E) \to At_F(E) \to T_F \to 0$$

Theorem 3.4. *If P is non-degenerate along Y, existence of a first order deformation is equivalent to the existence of a k-linear splitting of the null foliation Atiyah sequence which is a $(1/2,1)$-connection. Furthermore, if α_2^X is symmetric then a second order deformation exists if and only if the splitting agrees with the bracket.*

Proof. The first part is a restatement of Theorem 1.1. By definition a splitting, γ, commutes with the bracket when $c(\gamma) = 0$. Since α_2^X is symmetric this happens if and only if there is a second order deformation. □

4. Curved DGLA on Hochschild complex

4.1. **L_∞-algebras.** In this section we define *strongly homotopy Lie algebras*, commonly known as *L_∞-algebras*. We give the definitions and results in the curved L_∞ case, for lack of a convenient reference.

Let A be a graded vector space over a commutative ring k (not necessarily a field) which contains the rational numbers as a subring. The homogenous elements of degree n are denoted by A^n. The *suspension* of graded vector space is the graded vector space, $A[1]$, such that $A[1]^n := A^{n+1}$. Consider the cofree coassociative cocommutative counital coaugmented coalgebra generated by $A[1]$ $S(A[1]) := \oplus_{n\geq 0} S^n(A[1])$ where $S^n(A[1]) := (\otimes^n A[1])^{\Sigma_n} \simeq (\wedge^n A)[n]$ i.e. the set of tensors which are invariant under the natural action of the symmetric group on n elements. Recall, a counital coalgebra is coaugmented if there exists a coalgebra morphism $\eta : k \to C(V)$. The notion of a dg-coalgebra morphism will be defined shortly. The coalgebra structure is given by

$$\Delta(a_1 \wedge \cdots \wedge a_n) = \sum_{i=1}^{n} \sum_{\sigma\in\Sigma_{i,n-i}} e(\sigma)(a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(i)}) \otimes (a_{\sigma(i+1)} \wedge \cdots \wedge a_{\sigma(n)})$$

where $\Sigma_{i,n-i}$ is the set of $(i, n-i)$-shuffles of Σ_n i.e. $\sigma \in \Sigma_n$ such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. The sign is determined by the Koszul rule.

A *curved L_∞-algebra structure* on A is a codifferential Q of degree 1 on $S(A[1])$ i.e. a linear map

$$Q : S(A[1]) \to S(A[1])[1]$$

such that $\Delta Q = (Q \otimes id)\Delta + (id \otimes Q)\Delta$ and $Q^2 = 0$. In other words $S(A[1])$ has the structure of a dg-coalgebra. Any coderivation is completely determined by the values on the cogenerators given by the composition

$$\ell_n : S^n(A[1]) \to S(A[1]) \xrightarrow{Q} S(A[1])[1] \to A[2]$$

for all $n \geq 0$. The $\{\ell_k\}_{k\geq 0}$ are known as *higher brackets*. The condition $Q^2 = 0$ implies an infinite set of quadratic equations that $\{\ell_n\}_{n\geq 0}$ must satisfy known as *higher Jacobi relations*. If Q agrees with the coaugmentation i.e. $Q\eta = 0$ we simply say A is an L_∞-algebra. In the curved case the quadratic relation implies $\ell_1^2 = \ell_2\ell_0$ which is nonzero in general hence cohomology is not defined. Furthermore, for an L_∞-algebra A we set $H^*(A) := H^*(A, \ell_1)$.

If $\ell_n = 0$ for $n \neq 2$ then A is a graded Lie algebra. A dg Lie algebra is an L_∞ algera with $\ell_n = 0$ for $n \neq 1, 2$. A *curved dg (CDG) Lie algebra* is an L_∞-algebra with $\ell_n = 0$ for $n \geq 3$. The quadratic relation implies $\ell_1\ell_0 = 0$, $\ell_1\ell_1 = \ell_2\ell_0$, ℓ_2 satisfies the Jacobi identity and ℓ_1 is a derivation with respect to ℓ_2.

Let $(\mathcal{C}_i, Q_i)$ be two dg-coalgebras a *dg-colagebra morphism* is a morphism of vector spaces $F : \mathcal{C}_1 \to \mathcal{C}_2$ which is equivariant with respect to the codifferentials i.e. $FQ_1 = Q_2F$. In then case when $\mathcal{C}_i = S(A_i[1])$ for a graded vector space A_i then such a morphism is determined by a sequence of maps $F_n : \wedge^n A_1 \to A_2[1-n]$ for $n \geq 0$ which satisfy an infinite set of equations coming from the compatibility with the codifferentials. The explicit formulae for a DGLA are in [19]. In this case F_1 is a morphism of Lie algebras only up to a homotopy. In particular the category of dg Lie algebras with dg Lie algebra morphisms is not a full subcategory of the L_∞ category.

An L_∞-morphism $F : (S(A_1[1]), Q_1) \to (S(A_2[1]), Q_2)$ is a *quasi-isomorphism* if its first component $F_1 : A_1 \to A_2$ is an isomorphism on cohomology. Here we are assuming $\ell_0 = 0$ so cohomology is defined. An important theorem due to Kadeishvili says an L_∞-algebra is quasi-isomorphic to its cohomology.

Theorem 4.1. [15] *There exists a quasi-isomorphism of L_∞-algebras $H^*(A) \to A$ which lifts the identity of $H^*(A)$.*

A dg Lie algebra A *formal* if the induced brackets ℓ_n on $H^*(A)$ are 0 for $n \geq 3$.

The zeroes of Q are solutions of the *Maurer-Cartan equation* and put $Zero(Q) := \mathcal{MC}(A)$. In terms of the higher brackets $b \in A^1$ is an element of $\mathcal{MC}(A)$ if and only if

$$\sum_{k=0}^{\infty} \frac{1}{k!} \ell_k(b, \ldots, b) = 0 \tag{4.1}$$

If A is a dg Lie algebra then (4.1) is the usual Maurer-Cartan equation i.e. $db + \frac{1}{2}[b, b] = 0$. A dg-coalgebra morphism $F : \mathcal{C}_1 \to \mathcal{C}_2$ induces a mapping on solutions of the Maurer-Cartan equation, $F_* : \mathcal{MC}(\mathcal{C}_1) \to \mathcal{MC}(\mathcal{C}_2)$ since it commutes with the codifferentials.

4.2. **Homotopy theory of L_∞-algebras.** The primary difficulty in dealing with curved L_∞-algebras is that quasi-isomorphism no longer has any meaning i.e. cohomology is no longer defined. A replacement is homotopy equivalence which is more general than quasi-isomorphism. We follow the terminology and exposition of [10] where the case of A_∞-algebras was worked out in detail but little needs to be changed for curved L_∞-algebras. The proofs though are contained in [11].

In the category of topological spaces the notion of homotopy is very well-known. In particular, a homotopy between two morphisms $f, f' : X \to Y$ is another morphism $H : [0,1] \times X \to Y$ which lives in the category of topological spaces. Motivated by this there is similar notion of a homotopy in the category of L_∞-algebras. But first we must make sense of the what it means to take the product an L_∞-algebra, A, with the unit interval. This is called a *model of* $[0,1] \times A$ in [11].

Definition 4.2. Define an L_∞-algebra $A[1] \otimes k[t, dt]$ where $A[t]$ is the polynomial ring with coefficients in A. An element of $A[1] \otimes k[t, dt]$ is written as a sum $a(t) + b(t)dt$ where $a(t), b(t) \in A[t]$. Also define $\deg dt = 1$ and set $\widetilde{\ell}_0(1) = \ell_0(1) + 0dt$. The higher brackets are given by

$$\widetilde{\ell}_1(a(t) + b(t)dt) = \ell_1(a(t)) - \ell_1(b(t))dt - \frac{db}{dt}dt$$

$$\begin{aligned}\widetilde{\ell}_k(a_1(t) + b_1(t)dt, \ldots, a_k(t) + b_k(t)dt) &= \ell_k(a_1(t), \ldots, a_k(t)) \\ &\quad + \sum_{j=1}^{k} (-1)^{|a_1| + \cdots + |a_{j-1}| + j} \ell_k(a_1(t), \ldots, b_j(t), \ldots, a_k(t))dt\end{aligned}$$

We define the L_∞ *evaluation homomorphism* $Eval_{t=t_0} : A[1] \otimes k[t, dt] \to A[1]$ as

$$Eval_{t=t_0}(a(t) + b(t)dt) = a(t_0)$$

for $t_0 \in \mathbb{R}$.

Definition 4.3. Two L_∞ morphisms $f, g : A \to A'$ are *homotopic*, denoted by $f \sim g$, if there exists an L_∞ homomorphism $H : A \to A' \otimes k[t, dt]$ such that $Eval_{t=0} \circ H = f$ and $Eval_{t=1} \circ H = g$.

Definition 4.4. An L_∞ morphism $f : A \to A'$ between L_∞-algebras is a *homotopy equivalence* if there exists an L_∞ morphism $g : A' \to A$ such that $fg \sim id$ and $gf \sim id$. Furthermore, we say A and A' are *homotopy equivalent* if there exists a homotopy equivalence as above.

If $\ell_0 = 0$ then a quasi-isomorphism is the same as a homotopy equivalence. In the category of L_∞-algebras with L_∞ homomorphisms a quasi-isomorphism has a homotopy inverse which is also a quasi-isomorphism. This is not true in the category of dg Lie algebras with dg Lie algebra homomorphisms as there exist quasi-isomorphisms without inverses. This is one of the reasons to enlarge the dg Lie algebra category to the homotopy L_∞-category.

We now introduce a covariant functor $\mathcal{MC}(A)$ from the category of Artin k-local algberas to the category of sets. Let $\mathcal{R}$ be such an algebra, which we consider as a graded algebra concentrated in degree 0, and $\mathfrak{m}_\mathcal{R}$ the maximal ideal. Since $\mathcal{R}$ is concentrated in degree 0 we have $(A \otimes \mathfrak{m}_\mathcal{R})^i = A^i \otimes \mathfrak{m}_\mathcal{R}$. Define the functor $\mathcal{MC}(A)(\mathcal{R}) := \mathcal{MC}(A \otimes \mathfrak{m}_\mathcal{R})$. If $\psi : \mathcal{R} \to \mathcal{R}'$ is a morphism of algebras and $b \in \mathcal{MC}(A \otimes \mathcal{R})$ then $(1 \otimes \psi)(b) \in \mathcal{MC}(A \otimes \mathfrak{m}_{\mathcal{R}'})$. Hence there is a morphism $\mathcal{MC}(A)(\mathcal{R}) \to \mathcal{MC}(A)(\mathcal{R}')$ which makes $\mathcal{MC}(A)$ into a covariant functor. However, the set $\mathcal{MC}(A)(\mathcal{R})$ is too large to be homotopy invariant so instead we look at equivalence classes in $\mathcal{MC}(A)(\mathcal{R})$.

Definition 4.5. Let $b, b' \in \mathcal{MC}(A)(\mathcal{R})$ then b and b' are *gauge equivalent* denoted by $b \sim b'$ if there exists an element $\widetilde{b} \in \mathcal{MC}(A \otimes k[t, dt])(\mathcal{R})$ such that $Eval_{t=0} \circ \widetilde{b} = b_0$, $Eval_{t=1} \circ \widetilde{b} = b_1$.

The proof that gauge equivalence is an equivalence relation is found in [11]. Using this define the *deformation set* as

$$Def(A)(\mathcal{R}) := \mathcal{MC}(A)(\mathcal{R})/\sim$$

For $\psi : \mathcal{R} \to \mathcal{R}'$ there is a morphism $\psi_* : \mathcal{MC}(A)(\mathcal{R}) \to \mathcal{MC}(A)(\mathcal{R}')$. Thus there is a *deformation functor* $Def(A)$ from algebras as above to the category of sets. The following theorem provides justification for taking gauge equivalence classes of solutions to the Maurer-Cartan equation.

Theorem 4.6. [10, Theorem 2.2.2] *If A is homotopy equivalent to A' then the deformation functor $Def(A)$ is equivalent to $Def(A')$.*

One can extend the above discussion to projective limits of Artin local algebras. In this paper we will consider the usual projective limit: $\epsilon k[[\epsilon]] = \varprojlim(\epsilon k[\epsilon]/\epsilon^r k[\epsilon])$ cf. [19].

By definition $\widetilde{b} = a(t) + b(t)dt \in \mathcal{MC}(A \otimes k[t, dt])(\mathcal{R})$ if and only if

$$\frac{da}{dt} + \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \ell_k(b(t), a(t), \ldots, a(t)) = 0$$

$$\sum_{k=0}^{\infty} \ell_k(a(t), \ldots, a(t)) = 0$$

For the first equation we used skew-symmetry of the ℓ_k. This implies $a(t) \in \mathcal{MC}(A)(\mathcal{R})$ for all t. If A is a dg Lie algebra gauge equivalence reduces to $da/dt = \ell_1(b) + \ell_2(b,a)$ cf. [19].

As noted above the difficulty in dealing with curved L_∞-algebras is cohomology is not defined. To overcome this we can *twist* by a Maurer-Cartan element to an L_∞-algebra with $\ell_0 = 0$. Suppose $b \in \mathcal{MC}(A)$ and define

$$\ell_k^b(a_1, \dots, a_k) = \sum_{j=0}^{\infty} \frac{1}{j!} \ell_{k+j}(\underbrace{b, \dots, b}_{j-\text{times}}, a_1, \dots, a_k)$$

In particular,

$$\ell_0^b(1) = \sum_{j=0}^{\infty} \frac{1}{j!} \ell_j(b, \dots, b) = 0$$

It is then a straightforward calculation to see (A, ℓ_k^b) is an L_∞-algebra which we call the *b-twist of A*.

Definition 4.7. The *b-twisted cohomology* of an L_∞-algebra A is

$$H_b^*(A) := H^*(A, \ell_1^b)$$

Proposition 4.8. [11, Proposition 4.3.16] *If $b_0 \sim b_1$ then there is a homotopy equivalence $f : (A, \ell_k^{b_0}) \to (A, \ell_k^{b_1})$. This implies b-twisted cohomology depends only on the gauge equivalence class of b.*

4.3. **Commutative Deformations.** Let A be a ring and E be an A-module the Hochschild complex, $\mathfrak{g}_E^n := Hom_k(A^{\otimes n} \otimes E, E)$, is a dg Lie algebra. The differential is given by

$$\begin{aligned} d_{Hoch}\alpha(a_1, \dots, a_{k+1}, e) &:= a_1\alpha(a_2, \dots, a_{k+1}, e) + \sum_{i=1}^{k} (-1)^i \alpha(a_1, \dots, a_i a_{i+1}, \dots, a_{k+1}, e) \\ &+ (-1)^{k+1}\alpha(a_1, \dots, a_k, a_{k+1}e) \end{aligned}$$

and the Lie bracket is

$$[\alpha_1, \alpha_2]_G := \alpha_1 \circ \alpha_2 - (-1)^{kl} \alpha_2 \circ \alpha_1$$

where $\alpha, \alpha_1 \in \mathfrak{g}^k$ and $\alpha_2 \in \mathfrak{g}^l$ and

$$\alpha_1 \circ \alpha_2(a_1, \dots, a_{k+l}, e) = \alpha_1(a_1, \dots, a_k, \alpha_2(a_{k+1}, \dots, a_{k+l}, e))$$

We will just write $\mathfrak{g}$ instead of $\mathfrak{g}_E$ when there is no confusion. It is the well known in this case that [23, Lemma 9.1.9]

$$H^*(\mathfrak{g}) = Ext_A^*(E, E)$$

In this section we do not assume Y is coisotropic. Let X be an affine scheme and Y a subvariety with a vector bundle E. A *commutative deformation* of E as a coherent $\mathcal{O}_X$-module is a flat deformation to an $\mathcal{O}_X$-module. The module structure is given by

$$a \star e = ae + \epsilon\alpha_1(a, e) + \epsilon^2\alpha_2(a, e) + \cdots \tag{4.2}$$

If we define $\alpha^E := \epsilon\alpha_1 + \epsilon^2\alpha_2 + \cdots \in \mathfrak{g}^1[[\epsilon]]$ then associativity of (4.2) is equivalent to the Maurer-Cartan equation in $\mathfrak{g}[[\epsilon]]$

$$d_{Hoch}\alpha^E + \frac{1}{2}[\alpha^E, \alpha^E]_G = a\alpha^E(b,e) - \alpha^E(ab,e) + \alpha^E(a,be) + \alpha^E(a,\alpha^E(b,e)) \tag{4.3}$$

Two solutions $\alpha^E, (\alpha^E)'$ are *gauge equivalent* denoted by $\alpha^E \sim (\alpha^E)'$ if there exists a $\phi \in \mathfrak{g}^0[[\epsilon]]$ such that

$$\phi(a \star e) = a \star' \phi(e) \tag{4.4}$$

which restricts to the identity modulo ϵ. Such a ϕ is of the form $\phi = id + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots$ and (4.4) is equivalent to

$$\phi_n(ae) + \alpha_n(a,e) + \sum_{j+k=n-1} \phi_j(\alpha_k(a,e)) = a\phi_n(e) + \alpha'_n(a,e) + \sum_{j+k=n-1} \alpha'_j(a,\phi_k(e)) \tag{4.5}$$

for all $n \geq 1$. It is then straightforward to check that gauge equivalence as defined above is equivalent to gauge equivalence defined in the previous subsection cf. [19, Section 3.2]. The main result of this section is the following formality theorem in this setting cf. [1, Conjecture 2.36]:

Theorem 4.9. *In the above setting the dg Lie algebra $(\mathfrak{g}, d_{Hoch}, [,])$ is quasi-isomorphic to the abelian Lie algebra $(\wedge^* N(E), 0)$ i.e. $\mathfrak{g}_E$ is formal.*

First we need a lemma to compute the cohomology of $\mathfrak{g}_E$.

Lemma 4.10. *Let X be a smooth affine variety and Y a subvariety with a vector bundle E. Then there is an isomorphism*

$$H^*(\mathfrak{g}) \simeq \wedge^* N(E)$$

Proof. By [23, Lemma 9.1.9]

$$\mathcal{H}^p(\mathfrak{g}) = \mathcal{E}xt^p_{\mathcal{O}_X}(E,E)$$

The rest of the lemma is a special case of a more general calculation found in [8]. They compute $Ext_{\mathcal{O}_X}(E_1, E_2)$ where E_i are vector bundles supported on possibly distinct subvarieties. In our case the calculation simplifies dramatically so we include it for completeness.

To calculate $\mathcal{E}xt^p_{\mathcal{O}_X}(E,E)$ for E a vector bundle we use the change of ring spectral sequence:

$$\iota_*\mathcal{E}xt^p_{\mathcal{O}_Y}(E, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_Y, E)) \Rightarrow \mathcal{E}xt^{p+q}_{\mathcal{O}_X}(E,E)$$

Since X is affine and E is locally free over Y the spectral sequence becomes

$$\mathcal{E}xt^p_{\mathcal{O}_X}(E,E) = \mathcal{H}om_{\mathcal{O}_Y}(E, \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_Y, E)) = E^* \otimes_{\mathcal{O}_Y} \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_Y, E)$$

A lemma is needed in order to calculate $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_Y, E)$

Lemma 4.11. $\mathcal{H}_k \boldsymbol{L}\iota^*\iota_*\mathcal{O}_Y = \wedge^k N^*_{Y/X}$

Proof. First notice

$$\iota_*\mathbf{L}\iota^*\iota_*\mathcal{O}_Y = \iota_*(\mathbf{L}\iota^*\iota_*\mathcal{O}_Y \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_Y} \mathcal{O}_Y) = \iota_*\mathcal{O}_Y \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \iota_*\mathcal{O}_Y \tag{4.6}$$

where the second equality is the projection formula. Since ι is a closed embedding the underived pullback of the direct image fixes the sheaf. This gives

$$\begin{aligned}\mathcal{H}_k\mathbf{L}\iota^*\iota_*\mathcal{O}_Y &= \iota^*\iota_*\mathcal{H}_k\mathbf{L}\iota^*\iota_*\mathcal{O}_Y \\ &= \iota^*\mathcal{H}_k\iota_*\mathbf{L}\iota^*\iota_*\mathcal{O}_Y \\ &= \iota^*\mathcal{H}_k(\iota_*\mathcal{O}_Y \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \iota_*\mathcal{O}_Y) \\ &= \wedge^k N^*_{Y/X}\end{aligned}$$

The first equality is the above remark about ι being a closed embedding, the second uses ι_* is an exact functor, third is (4.6) from above. The last equality uses the well known fact that $\iota^* Tor_k^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) = \wedge^k N^*_{Y/X}$. □

Returning to the original calculation

$$\begin{aligned}\mathcal{E}xt^k_{\mathcal{O}_X}(\iota_*\mathcal{O}_Y, \iota_*E) &= \mathcal{H}^k\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_Y, \iota_*E) \\ &= \mathcal{H}^k\iota_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathbf{L}\iota^*\iota_*\mathcal{O}_Y, E) \\ &= \iota_*\mathcal{H}^k\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathbf{L}\iota^*\iota_*\mathcal{O}_Y, E)\end{aligned}$$

The second equality is the adjoint relation $\mathbf{L}\iota^* \dashv \iota_*$ [13, 2.5.10] and the last equality is exactness of ι_*. The Grothendieck spectral sequence in this case yields

$$\mathcal{H}^p\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{H}_q\mathbf{L}\iota^*\iota_*\mathcal{O}_Y, E) \Rightarrow \mathcal{H}^{p+q}\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathbf{L}\iota^*\iota_*\mathcal{O}_Y, E)$$

By the above

$$\iota_*\mathcal{H}^p\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{H}_q\mathbf{L}\iota^*\iota_*\mathcal{O}_Y, E) = \iota_*\mathcal{H}^p\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\wedge^q N^*_{Y/X}, E) = \iota_*\mathcal{H}^p(Y, \wedge^q N_{Y/X} \otimes_{\mathcal{O}_Y} E)$$

This implies since Y has no higher cohomology that

$$\mathcal{E}xt^k_{\mathcal{O}_X}(\iota_*\mathcal{O}_Y, \iota_*E) = \iota_*\mathcal{H}^0(Y, \wedge^k N_{Y/X} \otimes_{\mathcal{O}_Y} E) = \iota_*(\wedge^k N_{Y/X} \otimes_{\mathcal{O}_Y} E)$$

□

Proof of Theorem 4.9. First construct a contraction from $\mathfrak{g} \xrightarrow{\pi} \wedge^* N(E)$ which exists since the ground ring contains the rational numbers as a subring. By comparing symmetry properties the map

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{[\cdot,\cdot]_G} \mathfrak{g} \xrightarrow{\pi} \wedge^* N(E)$$

is identically 0. Now use the arguments from [14, Section 2] to see that $\wedge^* N(E)$ has no higher brackets. This is the definition that $\mathfrak{g}$ is formal. □

Applying Theorem 4.6 and using that a homotopy equivalence is the same as a quasi-isomorphism we get the following corollary

Corollary 4.12. *There is a bijection between commutative deformations up to equivalence and section of $N(E)$.*

4.4. **Noncommutative Deformations.** Once Poisson structures are introduced the problem is far more elaborate. Let X be an affine Poisson variety with Poisson bivector $P \in \Gamma(X, \wedge^2 T_X)$. A deformation quantization of $\mathcal{O}_X$ is an associative product of the form

$$a \star b = ab + \epsilon\alpha_1^X(a,b) + \epsilon^2\alpha_2^X(a,b) + \cdots$$

where $a, b \in \mathcal{O}_X$ and α_i^X are bi-differential operators. Associativity of $\star$ is equivalent to

$$a\alpha^X(b,c) - \alpha^X(ab,c) + \alpha^X(a,bc) - \alpha^X(a,b)c - \alpha^X(\alpha^X(a,b),c) + \alpha^X(a,\alpha^X(b,c)) = 0 \tag{4.7}$$

where $\alpha^X := \epsilon\alpha_1^X + \epsilon^2\alpha_2^X + \cdots$ and $a, b, c \in \mathcal{O}_X$. Equation (4.7) is the Maurer-Cartan equation in the Hochschild complex of $\mathcal{O}_X$ with the usual Hoschschild differential and Gerstenhaber bracket [19].

Given a subvariety $Y \subset X$ and a vector bundle E on Y which we view as a coherent $\mathcal{O}_X$-module define a quantization of E as a flat coherent $\mathcal{A}$-module, $\mathcal{E}$. Immediately from the above Y must coisotropic. The module action is still given by (4.2) but associativity of the action is

$$a\alpha^E(b,e) - \alpha^E(ab,e) + \alpha^E(a,be) - \alpha^X(a,b)e - \alpha^E(\alpha^X(a,b),e) + \alpha^E(a,\alpha^E(b,e)) = 0$$

We define gauge equivalence as in (4.4) which is still equivalent to (4.5) for all $n \geq 1$.

Unlike the commutative case, noncommutative deformations are not governed by a dg Lie algebra but a curved dg Lie algebra. Define

(1) $\ell_0(1) := -\alpha_X \otimes id_E$
(2) $\ell_1(\alpha)(a_1, \ldots, a_{k+1}, e) := d_{Hoch}\alpha(a_1, \ldots, a_{k+1}, e) + \sum_{j=1}^{k}(-1)^j\alpha(a_1, \ldots, \alpha_X(a_j, a_{j+1}), a_{j+2}, \ldots, a_{k+1}, e)$
(3) $\ell_2(\alpha_1, \alpha_2) := [\alpha_1, \alpha_2]_G$

The fact these make $\mathfrak{g}$ into a curved dg Lie algebra is a straightforward computation. In particular, $\ell_1(\ell_0(1)) = 0$ is precisely the Maurer-Cartan equation in $C^*(\mathcal{O}_X, \mathcal{O}_X)[[\epsilon]]$. Associativity of (4.2) is then given by solutions of the Maurer-Cartan equation

$$\ell_0(1) + \ell_1(\alpha) + \frac{1}{2}\ell_2(\alpha, \alpha) = 0 \tag{4.8}$$

That gauge equivalence is equivalent to that defined in section 4.2 follows since $\ell_1(\beta) = d_{Hoch}(\beta)$ for $\beta \in \mathfrak{g}^0$. Hence the deformation space controls noncommutative module deformations up to gauge equivalence.

Given a solution, α, of (4.8) define a deformed derivation by $\ell_1^\alpha(\beta) := \ell_1(\beta) + \ell_2(\alpha, \beta)$. It is easy to check that $\ell_1^\alpha\ell_1^\alpha = 0$ is equivalent to (4.8) so ℓ_1^α defines a differential on $\mathfrak{g}[[\epsilon]]$. There is a deformed dg Lie algebra structure on $\mathfrak{g}[[\epsilon]]$ given by (ℓ_1^α, ℓ_2).

Definition 4.13. Let E be a vector bundle on Y which has a deformation quantization, α^E. The *Poisson-Hochschild cohomology of* E is defined as

$$HP^*(Y, E, \alpha^E) := H^*(\mathfrak{g}[[\epsilon]], \ell_1^{\alpha^E}) \tag{4.9}$$

When P is nondegenerate and Y is Lagrangian with a line bundle L the "classical" limit of $HP^*(Y, L, \alpha^L)$ recovers the de Rham cohomology of Y. In a subsequent paper we will discuss the construction of a category consisting of pairs (Y, E) where Y is a coisotropic subvariety and E is a vector bundle supported on Y which has a deformation quantization. The endomorphisms will be the Poisson-Hochschild cohomology of E cf. [9, 17, 22].

Appendix A.

A.1. **Local equations for deformations.** Here we collect the standard formulas describing a second order deformation $\mathcal{A}_2$. The first two formulas must hold on each open subset U_i of an affine covering. To unload notation we write α_2^X instead of α_2^{Xi}.

$$a\alpha_1^X(b,c) - \alpha_1^X(ab,c) + \alpha_1^X(a,bc) - \alpha_1^X(a,b)c = 0 \tag{A.1}$$

$$a\alpha_2^X(b,c) - \alpha_2^X(ab,c) + \alpha_2^X(a,bc) - \alpha_2^X(a,b)c = \alpha_1^X(\alpha_1^X(a,b),c) - \alpha_1^X(a,\alpha_1^X(b,c)) \tag{A.2}$$

The next two formulas must hold on each double intersection $U_i \cap U_j$; we write β_1^X and β_2^X instead of β_1^{Xij} and β_2^{Xij}, respectively.

$$\beta_1^X(ab) - a\beta_1^X(b) - \beta_1^X(a)b = 0 \tag{A.3}$$

$$\begin{aligned}\beta_2^X(ab) - a\beta_2^X(b) - \beta_2^X(a)b &= \\ = \alpha_2^{Xj}(a,b) - \alpha_2^{Xi}(a,b) + \beta_1^X(a)\beta_1^X(b) &- \beta_1^X(\alpha_1^X(a,b)) + \alpha_1^X(\beta_1^X(a),b) + \alpha_1^X(a,\beta_1^X(b))\end{aligned} \tag{A.4}$$

We also give similar equations for the module action:

$$a\alpha_1(b,e) - \alpha_1(ab,e) + \alpha_1(a,be) = \alpha_1^X(a,b)e \tag{A.5}$$

$$a\alpha_2(b,e) - \alpha_2(ab,e) + \alpha_2(a,be) = \alpha_2^X(a,b)e + \alpha_1(\alpha_1^X(a,b),e) - \alpha_1(a,\alpha_1(b,e)) \tag{A.6}$$

$$\beta_1(ae) - a\beta_1(e) = \alpha_1^j(a,e) - \alpha_1^i(a,e) + \beta_1^X(a)e \tag{A.7}$$

$$\begin{aligned}&\beta_2(ae) - a\beta_2(e) = \\ &= \alpha_2^j(a,e) - \alpha_2^i(a,e) + \beta_2^X(a)e + \alpha_1^j(a,\beta_1(e)) - \beta_1(\alpha_1^i(a,e))) + \beta_1^X(a)\beta_1(e) + \alpha_1^j(\beta_1^X(a),e)\end{aligned} \tag{A.8}$$

In addition, there should be equalities on the triple intersections $U_i \cap U_j \cap U_k$ saying that the transition functions satisfy the cocycle condition. Only the module version of these equations is relevant to this paper:

$$\beta_1^{kj} + \beta_1^{ji} - \beta_1^{ki} = 0; \quad \beta_2^{kj} + \beta_2^{ji} - \beta_2^{ki} = \beta_1^{kj} \circ \beta_1^{ji}$$

However, we will avoid dealing with these equations directly by assuming that $H^2(Y, \mathcal{O}_Y(E)) = 0$. In our applications the difference $LHS - RHS$ is always $\mathcal{O}_Y$-linear and satisfies the cocycle condition on the fourfold intersections. Since the 2-cocycle can always be resolved due to the assumption, we can adjust β_*^{ji} to ensure that the last two equations hold as well.

Lemma A.1. *Let A be the ring of regular functions on a smooth affine variety X and E a projective module of finite rank over the quotient ring B corresponding to a smooth affine subvariety Y. Let $R : A \otimes_k E \to E$ be a k-linear map. Then $\beta(ae) - a\beta(e) = R(a,e)$ for some $\beta \in Hom_k(E,E)$ if and only if R vanishes in $I \otimes_k E$ and also satisfies*

$$aR(b,e) - R(ab,e) + R(a,be) = 0$$

Similarly, if $G : A \otimes_k A \otimes_k E \to E$ is a k-linear map then $a\rho(b,e) - \rho(ab,e) + \rho(a,be) = G(a,b,e)$ for some $\rho : A \otimes_k E \to E$ if and only if the restriction of G to $I \otimes_k I \otimes_k E \to E$ is symmetric in the first two arguments and

$$aG(b,c,e) - G(ab,c,e) + G(a,bc,e) - G(a,b,ce) = 0$$

Moreover, if R, resp. G is an algebraic differential operator in each of its arguments then one can choose β, resp. ρ, with the same property.

Lemma A.2. *Let $\beta \in \mathfrak{g}_E^i$ (see section 4.3 for the notation) be a cocycle for $i = 2, 3$ then the antisymmetrization of β when restricted to I_Y is $\mathcal{O}_X$ polylinear.*

Proof. The conclusion for $i = 2$ is clear. For $i = 3$ we have

$$a\beta(b, c, e) - \beta(ab, c, e) + \beta(a, bc, e) - \beta(a, b, ce) = 0$$

for all $a, b, c \in A$. Hence for $a \in A, x, y \in I$

$$\begin{aligned} a\beta(x, y, e) - a\beta(y, x, e) - \beta(ax, y, e) + \beta(y, ax, e) &= -\beta(a, xy, e) - a\beta(y, x, e) + \beta(ay, x, e) \\ &= -\beta(a, xy, e) + \beta(a, yx, e) = 0 \end{aligned}$$

and

$$\begin{aligned} &a\beta(x, y, e) - a\beta(y, x, e) - \beta(x, y, ae) + \beta(y, x, ae) \\ &\quad = a\beta(x, y, e) - a\beta(y, x, e) + \beta(xy, a, e) - \beta(x, ya, e) + \beta(y, x, ae) \\ &\quad = a\beta(x, y, e) - a\beta(y, x, e) + \beta(xy, a, e) - \beta(x, ya, e) - \beta(yx, a, e) + \beta(y, xa, e) \\ &\quad = a\beta(x, y, e) - a\beta(y, x, e) - \beta(ax, y, e) + \beta(ay, x, e) \\ &\quad = -\beta(a, xy, e) + \beta(a, yx, e) = 0 \end{aligned}$$

□

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